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Average Cross Sections for the $C^{12}(C^{12},\alpha)Ne^{20}$ Reactions

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Measurements of the integrated cross sections and angular distributions of the $C^{12}(C^{12},\alpha)Ne^{20}$ reactions, for alpha particles leaving Ne^{20} in the ground state and first excited state, are averaged over the energy interval of 10.15 to 12.8 MeV (in the center-of-mass frame) and compared with calculations based on the statistical theory of nuclear reactions. The calculations employ optical-model transmission functions of the correct energy for each of the channels open to the compound nucleus. The applicability of the statistical theory to heavy-ion reactions is discussed. Quantitative estimates are made of the random error in calculated cross sections arising from the finite size of the averaging interval and the cross-section fluctuations in the interval. Generally good agreement is obtained between calculated and measured average cross sections for both the magnitude of integrated cross sections and the shape of angular distributions. The statistical theory calculations are combined with experimental values of total level widths (found in a parallel paper discussing the cross-section fluctuations underlying our average cross sections) to yield level spacings in Mg^{24}, at excitation energies of 20–25 MeV. Individual level spacings for states with $J\Pi=2^+, 4^+, 8^+$ lead to estimates of the spin cutoff parameter. The corresponding value of the moment of inertia for the highly excited states of Mg^{24} is considerably larger than that of the ground-state band.

1. INTRODUCTION

FOR several decades, formulas based on the compound-nucleus picture have been employed to describe the gross features of average cross sections for heavy-ion reactions. More recently, the average cross-section theory has evolved into a useful quantitative tool and the theory of cross-section fluctuations has been developed to describe the cross-section structure underlying the average. In the present paper we employ the modern statistical theory of nuclear reactions to obtain quantitative estimates of the average cross sections of the $C^{12}(C^{12},\alpha)Ne^{20}$ reactions, for alpha-particle emission to the ground state and first excited state of Ne^{20}. The experiments, described in a parallel paper,[1] were made at center-of-mass energies between 10.15 and 12.8 MeV. The parallel paper also discusses the description of the cross-section structure with the theory of fluctuations. The results of the two papers complement each other: On the one hand, the estimates of average cross sections for individual partial waves yield the weight coefficients required in the preceding paper to compare observed fluctuations with theory; on the other hand, the analysis of fluctuations enables us here to estimate the effect of the finite size of the averaging interval.

2. THEORY OF AVERAGE CROSS SECTIONS

In the theory of average cross sections[2] the angle-integrated cross section $\bar{\sigma}_{\alpha\alpha'}$ averaged over compound nucleus fluctuations may be written

$$\bar{\sigma}_{\alpha\alpha'}=(\pi/k_\alpha^2)\sum_{J\Pi}[(2J+1)/(2I+1)(2i+1)] \times\{\sum_{sl}T_l(\alpha)\}\left\{\frac{\sum_{s'l'}T_{l'}(\alpha')}{\sum_{c''}T_{l''}(c'')}\right\},\quad (1)$$

where unprimed quantities refer to the incoming channel c, primed quantities to the outgoing channel c', and the sum in the denominator runs over all possible outgoing channels. The quantum numbers of each channel c are $c=(\alpha,I,i,s,l,J,M_J,\Pi)$ where α labels the pair of particles and their state of excitation, I and i are intrinsic spins, s is the channel spin ($\mathbf{s}=\mathbf{I}+\mathbf{i}$), l the orbital angular momentum, J the total angular momentum ($\mathbf{J}=\mathbf{l}+\mathbf{s}$), M_J its z component (assumed to be averaged over and ignored) and Π the total parity. In (1), k_α is the wave number of the incident channel. The transmission functions $T_l(\alpha)$ of (1) may be calculated from the complex phase shifts δ_l of the optical-model potential appropriate to each pair of particles

$$T_l=1-|e^{2i\delta_l}|^2 \approx\tau_l/(1+\tau_l/4)^2 \approx\tau_l\ (\tau_l\ll1),\quad (2)$$

with

$$\tau_l\equiv2\pi\langle\Gamma_l{}^{J\Pi}\rangle/D^{J\Pi},\quad (2a)$$

where the approximate connection between transmission functions on the one hand, and average partial widths and level spacings on the other, follows from nuclear reaction theory[2,3] (see discussion below). If the optical-model potential contains a spin-orbit term, then the phase shifts and transmission functions depend on the

[1] E. Almqvist, J. Kuehner, D. McPherson, and E. Vogt, Phys. Rev. **136**, B86 (1964), preceding paper.

[2] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958). See also E. W. Vogt, Rev. Mod. Phys. **34**, 723 (1962).

[3] The connection between T and τ of Eq. (2) is two-valued. For $\tau=4$, $T=1$, but for any other value of T, τ has two possible values. To resolve any ambiguity we note that for a black nucleus τ increases to the value 4 as the penetrability approaches its maximum value $P\rightarrow kR$, where R is the nuclear radius. The higher values of τ are not possible for a black nucleus. For an optical potential at low energy, the transmission functions are very small and τ might be very small or very large; however, the penetrability makes τ small so that no ambiguity exists. At high energies, τ can exceed the value 4 in the vicinity of a giant resonance so that the connection (2) is both approximate and uncertain. We use the connection mostly for heavy ions at moderate energies with large imaginary potentials which remove giant resonances. Hence, we can quite safely restrict the value of τ to be less than 4 thus making the connection (2) single-valued.

channel spin s. For the reactions we are considering, such spin-orbit terms are unimportant and the transmission functions are then equal for all channel spins allowed by angular momentum conservation ($\mathbf{J}=\mathbf{l}+\mathbf{s}$).

The expression for the average angular distribution corresponding to (1) is

$$\frac{d\bar{\sigma}_{\alpha\alpha'}}{d\Omega}=\sum_L \frac{1}{4k^2}\sum_{J\Pi}\frac{1}{(2I+1)(2i+1)}\{\sum_{sl}T_l(\alpha)\}\times\sum_{s'l'}\left\{\frac{T_{l'}(\alpha')}{\sum_{c''}T_{l''}(\alpha'')}\right\}Z(lJlJ;sL)\times Z(l'Jl'J;s'L)(-)^{s-s'}P_L(\cos\theta), \quad (3)$$

where the Z's are the usual Z coefficients and the $P_L(\cos\theta)$ are Legendre polynomials. θ is the angle of the outgoing particles relative to the incident beam.

The use of the formula (1) for our heavy-ion reactions is more fashionable than theoretical. On the one hand, the result (1) follows from a combination of fundamental conservation laws with optical-model concepts without direct introduction of the compound nucleus; on the other hand, it follows from proper theories of the compound nucleus only in rather simple cases. We discuss both derivations to understand the applicability of the statistical theory to heavy ion reactions.

The result (1) for the average integrated cross section $\bar{\sigma}_{\alpha\alpha'}$ follows directly from the following assumptions:

(A) Total angular momentum J and parity Π are conserved allowing us to write

$$\bar{\sigma}_{\alpha\alpha'}\equiv\sum_{J\Pi}\bar{\sigma}_{\alpha\alpha'}{}^{J\Pi}. \quad (4)$$

(B) Each $\bar{\sigma}_{\alpha\alpha'}{}^{J\Pi}$ may be written as a product of two factors

$$\bar{\sigma}_{\alpha\alpha'}{}^{J\Pi}\equiv[\sigma_\alpha{}^{J\Pi}(\text{comp})][T_{\alpha'}{}^{J\Pi}/\sum_{c''}T_{c''}{}^{J\Pi}], \quad (5)$$

where the first factor is a compound-nucleus formation cross section, depending only on the incoming channel α. The second factor is a branching ratio whose numerator involves only the outgoing channel α' and whose denominator involves a sum over all possible reaction channels. (The $T_\alpha{}^{J\Pi}$ will be identified with transmission functions below but for the moment we regard them merely as branching ratio quantities.)

(C) The average cross section $\bar{\sigma}_{\alpha\alpha'}{}^{J\Pi}$ obeys a reciprocity theorem of the following kind:

$$k_\alpha{}^2\bar{\sigma}_{\alpha\alpha'}{}^{J\Pi}=k_{\alpha'}{}^2\bar{\sigma}_{\alpha'\alpha}{}^{J\Pi}. \quad (6)$$

Equations (6) and (5) lead at once to a connection between the branching ratio and the compound-nucleus formation cross section of the corresponding channel. The connection is

$$[T_{\alpha'}{}^{J\Pi}/\sum_{c''}T_{c''}{}^{J\Pi}]=k_{\alpha'}{}^2\sigma_{\alpha'}{}^{J\Pi}(\text{comp})/\sum_{c''}k_{c''}{}^2\sigma_{c''}{}^{J\Pi}(\text{comp}). \quad (7)$$

(D) For each channel, the compound-nucleus formation cross section is equal to the absorption cross section $\sigma_\alpha(\text{abs})$ of the optical-model potential for the channel. If the complex phase shifts of the optical-model potential are δ_l, then

$$\sigma_\alpha(\text{abs})=\frac{\pi}{k^2}\sum_l(2l+1)(1-|e^{2i\delta_l}|^2)=\sum_{J\Pi}\frac{\pi}{k^2}\frac{(2J+1)}{(2I+1)(2i+1)}\sum_{l=J-s}^{J+s}\sum_{s=|I-i|}^{I+i}(1-|e^{2i\delta_l}|^2)\equiv\sum_{J\Pi}\sigma_\alpha{}^{J\Pi}(\text{comp}). \quad (8)$$

The conservation of parity is ensured by having the total parity Π equal to the product of $(-)^l$ and the parity of the two particles in the channel. Use of (8) in (7) leads to

$$T_\alpha{}^{J\Pi}=\sum_{sl}T_l{}^{J\Pi}(\alpha), \quad (9)$$
$$T_l{}^{J\Pi}(\alpha)=1-|e^{2i\delta_l}|^2 \quad [\text{cf. (2), above}]$$

for J, l, s, Π satisfying conservation of total angular momentum and parity, and $T_l{}^{J\Pi}=0$ otherwise. [The superscripts $J\Pi$ on the transmission functions may be suppressed as in (1) and (3) if these conservation laws are understood.] Then we obtain, from (9), (8), and (5), the desired result (1) for the average integrated cross section $\bar{\sigma}_{\alpha\alpha'}$ of the statistical theory. The similar result (3) for the angular distribution is obtained in the same way with the additional assumption that incoming waves (and outgoing waves) of different orbital angular momentum do not, on the average, interfere with each other.

The derivation of the average cross sections just given is plausible but not proper—both the assumptions (B) and (D) are not obviously the result of any theory. The factorization of the cross section as in (5) holds near the peak of an isolated resonance as can be shown with the Breit-Wigner formula: It does not hold, in general, for interfering resonances. The situations with which we are dealing involve overlapping compound levels whose lifetime is comparable to the transit time of a nucleon across the nucleus. It is therefore of some importance to establish the statistical formulas with a proper theory of nuclear reactions.

The theory of nuclear reactions begins with the collision matrix $U_{cc'}$ in terms of which the reaction cross section $\sigma_{\alpha\alpha'}$ (integrated over all angles and averaged over channel spins and orbital angular momenta) may be written

$$\sigma_{\alpha\alpha'}=\frac{\pi}{k_\alpha{}^2}\sum_{J\Pi sls'l'}\frac{(2J+1)}{(2I+1)(2i+1)}|\delta_{cc'}-U_{cc'}|^2. \quad (10)$$

Equation (10) itself does not involve any nuclear reaction theory—it involves only the geometrical considerations arising in the connection between the cross section and the asymptotic form of the wave function.

The nuclear theory enters into the collision matrix. In the Wigner and Eisenbud theory[4] the collision matrix is written[2]

$$U_{cc'}=e^{i(\Omega_c+\Omega_{c'})}[\delta_{cc'}+i\sum_{\lambda\lambda'}\Gamma_{\lambda c}^{1/2}\Gamma_{\lambda' c'}^{1/2}A_{\lambda\lambda'}], \quad (11)$$

where

$$(\Gamma_{\lambda c})^{1/2}\equiv(2P_c)^{1/2}\gamma_{\lambda c}, \quad (12)$$

$$(A^{-1})_{\lambda\lambda'}=(E_\lambda-E)\delta_{\lambda\lambda'}-\sum_{c''}(S_{c''}+iP_{c''})\gamma_{\lambda c''}\gamma_{\lambda' c''}. \quad (13)$$

The Ω_c are potential scattering phase shifts (unimportant in reactions, $c\neq c'$), the $\Gamma_{\lambda c}$ are partial widths which in turn are products of reduced widths, $\gamma_{\lambda c}^2$, and penetration factors, $2P_c$. The shift functions S_c depend on the wave functions of the channels c and on the boundary condition numbers b_c at each channel radius which define the compound states. The most natural choice of the boundary condition numbers is that which makes each shift function vanish at the energy E of the reactions.[2] The E_λ are the characteristic energies of the compound states. Retention of only one compound state in (11) leads at once to the familiar Breit-Wigner formula. For overlapping levels we must take many levels into account.

For the theory of average cross sections a useful approximation to (11) is obtained by expanding the level matrix A about its diagonal. The result for $U_{cc'}$ is [apart from phases unnecessary for $|U_{cc'}|^2$ as in (10)]

$$U_{cc'}\approx\sum_\lambda\frac{\Gamma_{\lambda c}^{1/2}\Gamma_{\lambda c'}^{1/2}}{E_\lambda-E-(i/2)\Gamma_\lambda}, \quad (14)$$

where we have retained only the first term in the expansion of A about its diagonal value. This approximation is valid if, for each channel, the following condition holds:

$$2\pi\langle\Gamma_{\lambda c}\rangle/D^{J\Pi}\ll 1, \quad (15)$$

where $\langle\Gamma_{\lambda c}\rangle$ is the average level width and $D^{J\Pi}$ the average spacing of the compound states whose total angular momentum and parity are $J\Pi$. The conditions under which (15) applies are discussed below. The result (14) makes the collision matrix a simple sum of Breit-Wigner amplitudes.

In order to obtain average cross sections with (14) and (10), we must obtain the average over a suitable energy interval of

$$|U_{cc'}|^2=4P_cP_{c'}\sum_{\lambda\lambda'}\times\frac{\gamma_{\lambda c}\gamma_{\lambda c'}\gamma_{\lambda' c}\gamma_{\lambda' c'}}{[E_\lambda-E-(i/2)\Gamma_\lambda][E_{\lambda'}-E+(i/2)\Gamma_{\lambda'}]}. \quad (16)$$

If we take the average of (16) over an energy interval δ containing many levels (if the total width, $\Gamma_\lambda\equiv\sum_c\Gamma_{\lambda c}$, is larger than the level spacing we assume $\delta\gg\Gamma_\lambda$), then we can ignore the terms in the double sum from levels outside δ because the denominators make such contributions small. We also assume that the energy interval δ is small enough that the penetrabilities P_c and $P_{c'}$ may be regarded as constants. The average of (16) is then found by restricting λ and λ' to the levels in δ, multiplying (16) by δ^{-1} and integrating over energy E from $-\infty$ to $+\infty$. The integral is evaluated by completing a contour in the upper half of the complex plane, and the result is $2\pi i$ times the sum of the residues of (16) at the poles located at $E=E_{\lambda'}+(i/2)\Gamma_{\lambda'}$

$$\langle|U_{cc'}|^2\rangle_{\text{av}}=\frac{4\pi P_cP_{c'}}{\delta}\times\sum_{\lambda\lambda'\text{ in }\delta}\frac{\gamma_{\lambda c}\gamma_{\lambda c'}\gamma_{\lambda' c}\gamma_{\lambda' c'}(\Gamma_\lambda+\Gamma_{\lambda'})}{(E_\lambda-E_{\lambda'})^2+\frac{1}{4}(\Gamma_\lambda+\Gamma_{\lambda'})^2}. \quad (17)$$

For each term in the sum over λ of (17) there will be about $\langle\Gamma_\lambda\rangle/D$ contributions from the λ' sum, and therefore about $\delta\langle\Gamma_\lambda\rangle/D^2$ pairs altogether. The reduced width amplitudes $\gamma_{\lambda c}$ are assumed to have random sign fluctuations so that the contribution from the terms with $\lambda\neq\lambda'$ will be proportional to the square root, $(\delta\langle\Gamma_\lambda\rangle/D^2)^{1/2}$, of the number of pairs while the corresponding contribution from the diagonal terms $\lambda=\lambda'$ is δ/D. With $\delta\gg\Gamma_\lambda$ we can neglect the cross product terms yielding

$$\langle|U_{cc'}|^2\rangle_{av}=(\delta/D)8(\pi/\delta)P_cP_{c'}[\langle\gamma_{\lambda c}^2\rangle\langle\gamma_{\lambda c'}^2\rangle/\langle\Gamma_\lambda\rangle] \quad (18)$$
$$=T_cT_{c'}/\sum_{c''}T_{c''},$$

where the transmission functions T_c are related to the average resonance parameters by

$$T_c\equiv 2\pi\langle\Gamma_{\lambda c}\rangle/D^{J\Pi}=4\pi P_cs_c, \quad (19)$$

in which s_c is the strength function.

Use of (19) in (11) yields at once the formula (1) for average cross sections except that the transmission functions now are given in terms of average resonance parameters rather than optical-model phase shifts. The correspondence between the behavior of nuclear transmission functions and those of the optical model is very close[2,5] and forms one of the principal justifications of the optical model.

The derivation of the average cross sections of the compound nucleus depends on the approximations (15) and on the assumption of random signs for the $\gamma_{\lambda c}$. The random sign approximation is also essential for the theory of fluctuations (see preceding paper[1]). The approximations (15) can be shown to be generally almost adequate for nucleon channels but completely inadequate for heavy ions and alpha particles. For most of the channels which we are considering the optical-model transmission functions are very close to

[4] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947). See also E. P. Wigner, *ibid.* **70**, 15 (1946); **70**, 606 (1946), T. Teichmann, *ibid.* **77**, 506 (1950); **83**, 141 (1952), T. Teichmann and E. P. Wigner, *ibid.* **87**, 123 (1952).

[5] A. M. Lane, R. G. Thomas, and E. P. Wigner, Phys. Rev. **98**, 693 (1955).

unity so that (15) does not hold. In the absence of a proper theory for the general case we discuss the simple case of one open channel. The results of this special case will be used for our more general connection between resonance parameters and optical-model transmission functions.

For the complex square well[2] of radius R the $\mathcal{R}$ function may be written

$$\mathcal{R}_l=\sum_p(\hbar^2/mR^2)/(E_p-E-iW) \equiv \mathcal{R}_l^\infty+i\pi s_l, \tag{20}$$

where the energies E_p belong to the single-particle levels in the real part of the complex square well and W is the imaginary part of the well depth. The corresponding transmission functions are found (from the absorption cross section) to be

$$T_l=4\pi P_l s_l/[(1-S_l\mathcal{R}_l^\infty+\pi P_l s_l)^2+(P_l\mathcal{R}_l^\infty+\pi S_l s_l)^2], \tag{21}$$

where the penetrabilities P_l and the shift functions S_l are the ones discussed above. The modifications of (21) caused by the diffuse-edge of the nuclear complex well are rather straightforward modifications[2] of P_l and S_l. If s_l is taken as the strength function—as seems appropriate from (20)—then (21) already gives a generalization of (19) which reduces to (19) when T_l is small.

The total cross section σ_T is linear in the diagonal elements of the collision matrix—a result following from flux conservation. In the case of only one channel the collision function has poles only in the lower half-plane. We may then follow the suggestion of Thomas[6] and average the cross section (or, equivalently, the collision function, U, since the cross section is linear in U) by moving the averaging interval δ up into the complex plane by an amount ϵ

$$\delta\gg\epsilon\gg D, \tag{22}$$

where D is the average spacing of the compound levels. The new average, with E replaced by $E+i\epsilon$, is equal to the old because the new interval can be connected to the old forming a closed contour containing no poles: If $\delta\gg\epsilon$ holds, the integral over the two connecting ends is negligible.

When E is replaced by $E+i\epsilon$ (with $\epsilon\gg D$ as assumed), then neither the collision function nor the $\mathcal{R}$ function have strong fluctuations. The $\mathcal{R}$ function of the nuclear reaction theory is then

$$\begin{aligned}\mathcal{R}(E+i\epsilon)&=\sum_\lambda \gamma_\lambda^2/(E_\lambda-E-i\epsilon)\\&=\int_{-\infty}^{\infty}\frac{\langle\gamma_\lambda^2\rangle}{D}(E_\lambda-E-i\epsilon)^{-1}dE_\lambda\\&=i\pi\frac{\langle\gamma_\lambda^2\rangle}{D}+P\int\frac{\langle\gamma_\lambda^2\rangle}{D}(E_\lambda-E)^{-1}dE_\lambda,\end{aligned} \tag{23}$$

[6] R. G. Thomas, Phys. Rev. 97, 224 (1955).

where P stands for the principal value of the improper integral. The important point is that the average $\mathcal{R}$ function, (23), has a real and imaginary part connected in exactly the same way as the $\mathcal{R}$ function (20) of the complex potential well. Moreover, the imaginary part of (23) is $i\pi s$, where s is the strength function and the s of (20) has the giant resonance behavior expected of nuclear strength functions.[5] Therefore, the average total cross section of the nucleus agrees with the total cross section of the complex potential well even when the transmission function approaches unity. For the one-channel case, then, the higher order corrections (in $4\pi P_l s_l$) of (21) are expected to correspond to those of the nucleus. If we choose the boundary condition numbers so that $S_l=0$, and if the complex potential well has a large W such that $\mathcal{R}_l^\infty$ is small, then (21) reduces to the approximate formula [second step of (2)] used in this paper to connect optical-model transmission functions to nuclear strength functions.[3]

For the case of many channels, the reaction cross sections (10) are not linear in the collision matrix components and we cannot displace the averaging interval without crossing poles. For this case no proper theory of average cross sections has been given which applies when the transmission functions approach unity. Because the statistical theory in the form (1) applies in the case of many channels and all $T_c\ll1$, and in the case of one channel with arbitrary T_c, it is not unreasonable to expect that it will also be found to apply in the more general case of many channels with arbitrary T_c. The more general case applies to the heavy-ion reactions which we are discussing, and it is found that the formulas of the statistical theory agree very well with the observed average cross sections.

3. EFFECT OF FINITE AVERAGING INTERVAL

An estimate of the energy interval required to obtain true average cross sections can be made by calculating the fluctuations about the average. The fluctuations of the compound-nucleus cross sections in the case of overlapping levels ($\Gamma_\lambda\gg D$) as discussed in the accompanying paper arise from the reaction theory through the same random sign fluctuations of the $\Gamma_{\lambda c}{}^{1/2}$ used in the derivation of average cross sections. Thus, (14) is a complex amplitude

$$U_{cc'}\approx\sum_\lambda\frac{\Gamma_{\lambda c}{}^{1/2}\Gamma_{\lambda c'}{}^{1/2}}{E_\lambda-E-i\Gamma_\lambda/2}=u_{cc'}+iv_{cc'}, \tag{24}$$

whose average square, $(\langle u_{cc'}{}^2\rangle_{av}+\langle v_{cc'}{}^2\rangle_{av})$ was evaluated by (18). If the numerators of (24) have random signs and if $u_{cc'}$ and $v_{cc'}$ then have Gaussian distributions about the average value zero,

$$\langle u_{cc'}\rangle_{av}=\langle v_{cc'}\rangle_{av}=0, \tag{25}$$

then both $\langle u_{cc'}{}^2\rangle_{av}$ and $\langle v_{cc'}{}^2\rangle_{av}$ have chi-squared distributions with one degree of freedom. To find the probability distribution of $\langle|U_{cc'}|^2\rangle_{\rm av}$ we must find the relative values of $\langle u_{cc'}^2\rangle_{\rm av}$ and $\langle v_{cc'}^2\rangle_{\rm av}$. From (24) we have

$$v_{cc'}^2=\tfrac{1}{4}\sum_{\lambda\lambda'}\frac{\Gamma_{\lambda c}^{1/2}\Gamma_{\lambda' c}^{1/2}\Gamma_{\lambda c'}^{1/2}\Gamma_{\lambda' c'}^{1/2}\Gamma_\lambda\Gamma_{\lambda'}}{[(E_\lambda-E)^2+\frac{1}{4}\Gamma_\lambda^2][(E_{\lambda'}-E)^2+\frac{1}{4}\Gamma_{\lambda'}^2]}. \quad (26)$$

To obtain the average of (26) over an energy interval δ ($\delta\gg\Gamma_\lambda$, D) we again restrict the sum in (26) to the levels in δ, multiply by δ^{-1} and integrate over E_λ from $-\infty$ to $+\infty$. As in (17) the contribution of the diagonal terms ($\lambda=\lambda'$) in (26) to the integral is larger than the contribution of the cross-product terms ($\lambda\neq\lambda'$). The former can be integrated directly yielding

$$\begin{aligned}\langle v_{cc'}^2\rangle_{\rm av}&=(\delta/D)(\pi/\delta)(\Gamma_{\lambda c}\Gamma_{\lambda c'}/\Gamma_\lambda)\\&=\tfrac{1}{2}\langle|U_{cc'}|^2\rangle_{\rm av}\\&=\langle u_{cc'}^2\rangle_{\rm av}.\end{aligned} \quad (27)$$

The equality of $\langle v_{cc'}^2\rangle_{\rm av}$ and $\langle u_{cc'}^2\rangle_{\rm av}$ means that $\langle|U_{cc'}|^2\rangle_{\rm av}$, which is the sum of the two, has a chi-squared distribution with two degrees of freedom.

It should be noted that if the total level width Γ_λ is not much larger than the average level spacing, then the fluctuations of $u_{cc'}$ and $v_{cc'}$ are not independent and the distribution of $|U_{cc'}|^2$ becomes closer to that of a chi-squared distribution with one degree of freedom. The study of the fluctuations of the $C^{12}(C^{12},\alpha)Ne^{20}$ reactions in the accompanying paper[1] showed no evidence of such an effect.

The averaging interval ΔE required to obtain true average cross sections must be considerably larger than the full width Γ_λ of the compound levels. In the discussion of fluctuations in the accompanying paper[1] the ratio $\Delta E/\Gamma_\lambda$ was referred to as the sample size S. The sample size S together with the basic distribution law (an exponential distribution, i.e., a chi-squared distribution with two degrees of freedom) of the fluctuating cross sections permits a quantitative estimate of the effect of the finite size of the averaging interval.

To distinguish finite-sample averages of cross sections from true averages, we shall write the former as $\langle\sigma\rangle_S$ and the latter as $\bar\sigma$ (or, equivalently, $\langle\sigma\rangle$) and we use the generic symbol σ to denote either $\sigma_{cc'}$ or $d\sigma_{cc'}/d\Omega$. In the case of overlapping levels each fluctuating cross section may be written as

$$\sigma=\sum_j C_j x_j, \quad (28)$$

where the x_j is a fluctuating quantity whose probability distribution $P(x_j)$ is

$$P(x_j)=e^{-x_j}, \quad (29)$$

and the C_j are weight coefficients calculated from the statistical theory of average cross sections. For example, if σ is the integrated cross section $\sigma_{cc'}$, then the index j of (28) runs over all combinations of the quantum numbers $J\Pi sls'l'$ and the value of C_j for each value of j is given by the corresponding term on the right side of (1). If σ is the differential cross section of the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction to the ground state of Ne^{20}, then the index j contains only one term and the corresponding value of C_j is given by the right-hand side of (3). For more general differential cross sections, the index j runs over all combinations of s, m_s, s', and $m_{s'}$, as discussed in Ref. 1, and the value of C_j is then given by Eq. (16) of Ref. 1.

The sample average of σ is

$$\langle\sigma\rangle_S=(1/S)\sum_{i=1}^S\sum_j C_j x_{ij}, \quad (30)$$

where each x_{ij} has the exponential distribution (29). The corresponding true averages of σ are then

$$\langle\langle\sigma\rangle_S\rangle=\bar\sigma=\sum_j C_j, \quad (31)$$

in agreement with (1) and/or (3).

The distribution of $\langle\sigma\rangle_S$ rapidly approaches a normal distribution as S becomes large. For example, if the sum over j contains only one term $C_j=C$, then $\langle\sigma\rangle_S$ is a chi-squared distribution of $2S$ degrees of freedom. This distribution has its maximum at $(S-1)C/S$, although the true average value of $\langle\sigma\rangle_S$ is still C. Moreover, if we choose a probable error ϵ_S so that the probability of finding $\langle\sigma\rangle_S/\bar\sigma$ within $1\pm\epsilon_S$ is 50%, then

$$\epsilon_S\equiv d_S S^{-1/2}, \quad (32)$$

where $d_\infty=0.6745$ (the value for a normal distribution) and $d_{10}=0.6714$, $d_{50}=0.6739$. Thus, for even fairly small values of the sample size, the tendency toward the normal distribution enables one to use the usual estimate of the probable error.

For cross sections, (28) with more than one weight coefficient C_j, we estimate the probable error of $\langle\sigma\rangle_S$ by means of the second moment of the distribution about its mean value (31). The result is

$$\epsilon_S\approx 0.6745S^{-1/2}(\sum_j C_j^2)^{1/2}/(\sum_j C_j). \quad (33)$$

The probable errors quoted in the remainder of the paper are based on (33) with weight coefficients obtained from the calculated average cross sections and with sample size values from the fluctuation analysis.[1]

4. CALCULATION OF AVERAGE CROSS SECTIONS FOR THE $C^{12}(C^{12},\alpha)Ne^{20}$ REACTION

The $C^{12}+C^{12}$ reactions involve identical bosons with zero spin. These facts simplify the average cross section formula above: Because of the zero spin, the incident channel spin is zero so that l and J [in (1) and (3)] are equal; because of the identity of the C^{12} nuclei all odd values of l are excluded, Π is positive, and (1) and (3) must each be multiplied by a factor of 2. The resulting form of (1) is

$$\bar\sigma_{cc'}=\frac{2\pi}{k^2}\sum_{{\rm even}\ J}(2J+1)T_{l=J}(c)\frac{\sum_{s'l'}T_{l'}(c')}{\sum_{c''}T_{l''}(c'')}, \quad (34)$$

with similar modifications in the average angular dis-

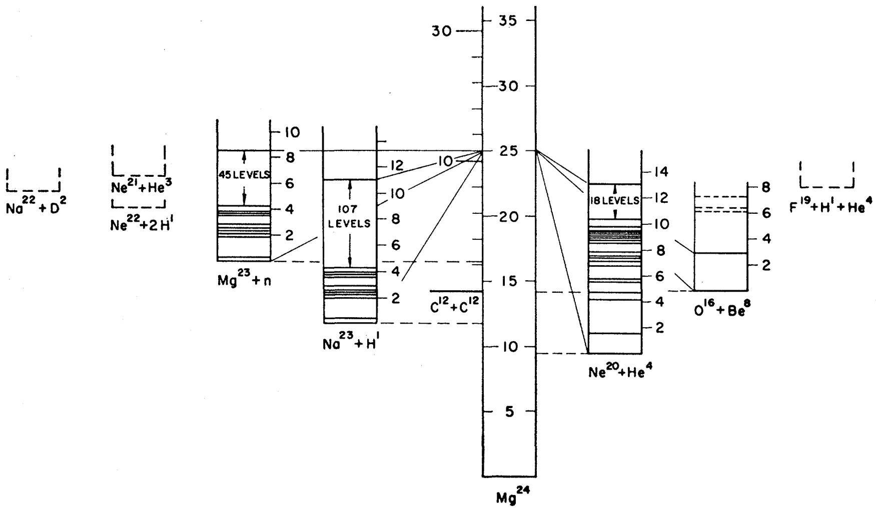

FIG. 1. The energy levels available for the decay of compound states of Mg^{24} at an excitation energy of $\sim$25 MeV. The computed cross sections for most of the final states shown are given in Table II.

tribution (3). The detailed calculations given below will show further simplification in that the cross sections which we are considering involve principally two values of l.

To make exact numerical calculations with the compound-nucleus theory requires knowledge of all the reaction channels which are open to a Mg^{24} compound nucleus at the appropriate excitation energy ($\sim$25 MeV in our case). Figure 1 shows the various final states to which the compound levels of Mg^{24} can decay for energies appropriate to the reaction under discussion. Fortunately, the spins and parities of most of the levels of Ne^{20} below about 12 MeV have recently become known.[7] The properties of most of the Na^{23} and Mg^{23} levels (involved in proton and neutron emission on Fig. 1) are presently not known, but it happens that these levels do not play a dominant role in the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction even though the nucleon emission channels far outnumber the levels of Ne^{20} available to α decay. The feature of the present problem which overrides other considerations is the large amount of angular momentum imparted to the compound system by the $C^{12}+C^{12}$ incident channel. The α particles carry off the angular momentum much more readily than nucleons. The relative unimportance of the nucleon-emission channels will be demonstrated by the statistical-theory calculations given below. The remaining open channels have, for the most part, known spins and parities enabling a proper statistical-theory calculation to be made.

To show the relative importance of nucleon- and alpha-particle channels in the decay of the Mg^{24} system, we give in Table II, the result of a statistical theory calculation carried out with (1) using all the final states shown on Fig. 1. For each final state the transmission functions were calculated from an optical model using parameters derived from fits to elastic-scattering data.[8] The optical-model parameters used in the calculation are given in Table I; the main results are quite insensitive to the optical-model parameters employed. The cross sections shown on the table (and in other tables below) refer to an energy of 11.4 MeV. Most of the partial cross sections are only moderately energy-dependent, so that the tabulated values can be taken as averages for the energy interval which we are considering.

The main difficulty with the statistical theory calculation of Table II is the lack of knowledge of the possible final states involved in nucleon emission. The estimates made in Table I and Fig. 1 about the number of such states is not subject to great uncertainty if one employs

TABLE I. The optical-model parameters employed in the statistical theory calculations. The optical potential is

$$V(r)=(-V_0+c_0E-iW_0)(1+e^{(r-R_0)/a})^{1} -i(W_g+c_1E)e^{-(r-R_0)^2/b^2}+V_c,$$

where V_c is the Coulomb potential and E the center-of-mass energy. The parameters were obtained from fits to elastic-scattering data for each of the projectiles.

System	V_0 MeV	R_0 F	W_0 MeV	a F	W_g MeV	b F	c_0	c_1
$C^{12}+C^{12}$	50	5.77	4.0	0.4	···	···	···	···
$Ne^{20}+\alpha$	45	5.0	10	0.5	···	···	···	···
$Na^{23}+p$	55	3.56	···	0.5	4.0	0.98	0.5	0.5
$Mg^{23}+n$	51	3.56	···	0.5	4.0	0.98	0.5	0.5
$O^{16}+Be^8$	50	5.77	4.0	0.4	···	···	···	···

[7] The levels of Ne^{20} used in the calculation include those in standard compilations and, in addition, a few others recently found by the Chalk River tandem group: J. A. Kuehner and J. D. Pearson, Can. J. Phys. **42**, 477 (1964); J. A. Kuehner and E. Almqvist, Bull. Am. Phys. Soc. **9**, 430 (1964); A. E. Litherland, C. Broude, and J. D. Pearson, *ibid.* **9**, 430 (1964). The new levels fill in gaps in the low-lying rotational bands of Ne^{20}.

[8] J. A. Kuehner and E. Almqvist, in *Proceedings of the Third Conference on Reactions Between Complex Nuclei, Asilomar California*, edited by A. Ghiroso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, 1963); Phys. Rev. **134**, B1229 (1964).

TABLE II. Results of the statistical theory calculation for the reaction cross sections of the $C^{12}+C^{12}$ system at a center-of-mass energy of 11.4 MeV. The first column gives the residual nucleus and its state of excitation, the second the spin and parity of each state, and the third the angle-integrated cross section $\sigma_{cc'}$ in mb for each final state. Bracketed energies and spins and parities refer to estimated values based on known level densities and spin distributions. For Na^{23} all the levels above 4.0 MeV should have bracketed energies and spins and parities. The properties of the levels of Mg^{23} are taken to be the same as those of the mirror nucleus Na^{23}. Levels sufficiently far below the Coulomb barrier (see Fig. 1) so that their cross sections are negligible are not listed in the table.

Residual state E^* in MeV	$J\Pi$	$\sigma_{cc'}$ mb
C^{12}		
ground	0+	13.5
4.43	2+	2.6
		Sum= 16.1
Be^8		
ground	0+	13.2
2.90	2+	11.4
		Sum= 24.6
Ne^{20}		
ground	0+	16.7
1.63	2+	41.2
4.25	4+	50.9
4.97	2−	15.4
5.63	3−	33.3
5.80	1−	12.9
6.75	0+	4.0
7.02	4−	24.0
7.19	3−	23.6
7.23	0+	3.5
7.46	2+	13.4
7.86	2+	11.8
8.52	5−	33.6
8.60	0+	2.2
8.70	1−	4.7
8.80	2+	8.5
8.84	6+	41.4
8.91	1−	4.3
9.10	4+	20.4
9.18	3−	12.2
9.58	2+	6.2
(10.30)	(2+)	4.4
(10.50)	(0+)	1.1
(10.50)	(1+)	0.8
(10.50)	(2+)	3.9
(10.50)	(3+)	3.3
(10.50)	(4+)	11.4
(10.50)	(2−)	1.7
(10.50)	(4−)	5.4
11.08	(0+)	0.7
11.16	(0−)	0.0
11.30	(1+)	0.5
11.39	(1−)	1.1
11.66	(2+)	1.6
11.84	(2−)	0.6
12.16	(3+)	0.6
12.24	(3−)	1.2
12.48	(4+)	1.5
13.06	(4−)	0.2
13.10	8+	2.7
		Sum=426.8
Na^{23}		
ground	3/2+	2.9
0.44	5/2+	4.1
2.08	7/2+	6.2
2.39	(1/2+)	0.8
2.64	(5/2+)	2.7
2.70	(3/2+)	1.8
2.98	(9/2+)	7.4
3.68	(3/2−)	3.1
3.85	(5/2−)	6.3
3.92	(3/2+)	1.5
4.30	3/2−	2.9
4.30	5/2+	2.0
4.30	7/2−	8.6
4.30	3/2+	1.4
4.30	5/2−	5.9
4.30	7/2+	4.0
4.50	1/2+	0.6
4.50	1/2−	1.5
5.50	1/2−	1.3
5.50	1/2+	0.5
5.80	3/2+	1.1
5.80	5/2−	4.3
5.80	7/2+	3.1
5.80	3/2−	2.2
5.80	5/2+	1.5
5.80	7/2−	6.2
6.20	1/2−	1.1
6.20	9/2+	4.0
6.20	1/2+	0.5
6.20	9/2−	11.7
6.80	3/2+	0.9
6.80	5/2−	3.1
6.80	7/2+	2.6
6.80	3/2−	1.6
6.80	5/2+	1.3
6.80	7/2−	4.4
7.60	3/2−	1.3
7.60	5/2+	1.1
7.60	7/2−	3.3
7.60	3/2+	0.8
7.60	5/2−	2.3
7.60	7/2+	2.1
7.70	1/2+	0.4
7.70	9/2−	6.7
7.70	11/2+	6.7
7.70	1/2−	0.7
7.70	9/2+	2.8
7.70	11/2−	9.9
8.30	3/2+	0.7
8.30	5/2−	1.8
8.30	7/2+	1.8
8.30	3/2−	1.0
Na^{23}		
8.30	5/2+	0.9
8.30	7/2−	2.5
8.50	1/2−	0.6
8.50	1/2+	0.3
8.80	1/2+	0.3
8.80	9/2−	4.3
8.80	11/2+	4.9
8.80	1/2−	0.5
8.80	9/2+	2.0
8.80	11/2−	6.3
9.00	3/2−	0.8
9.00	5/2+	0.7
9.00	7/2−	1.8
9.00	3/2+	0.5
9.00	5/2−	1.4
9.00	7/2+	1.4
9.04	3/2+	0.5
9.08	5/2−	1.3
9.12	7/2+	1.4
9.16	3/2−	0.8
9.20	13/2−	11.2
9.24	5/2+	0.6
9.28	7/2−	1.6
9.32	1/2−	0.4
9.36	9/2+	1.6
9.40	1/2+	0.2
9.44	9/2−	3.2
9.48	3/2+	0.5
9.52	5/2−	1.1
9.56	7/2+	1.1
9.60	3/2−	0.6
9.64	5/2+	0.5
9.68	7/2−	1.3
9.72	3/2−	0.6
9.76	5/2+	0.5
9.80	7/2−	1.2
9.84	3/2+	0.4
9.88	5/2−	0.9
9.92	7/2+	1.0
9.96	1/2+	0.2
10.00	9/2−	2.3
10.04	11/2+	2.9
10.08	13/2+	3.7
10.12	1/2−	0.3
10.16	9/2+	1.1
10.20	11/2−	2.9
10.24	3/2+	0.3
10.28	5/2−	0.7
10.32	7/2+	0.8
10.36	3/2−	0.4
10.40	5/2+	0.4
10.44	7/2−	0.8

TABLE II (*Continued*)

Residual state E^* in MeV	$J\Pi$	$\sigma_{cc'}$ mb	Residual state E^* in MeV	$J\Pi$	$\sigma_{cc'}$ mb	Residual state E^* in MeV	$J\Pi$	$\sigma_{cc'}$ mb
Na^{23}			Mg^{23}			Mg^{23}		
10.48	$\frac{1}{2}-$	0.2	3.85	$\frac{5}{2}-$	4.9	6.80	$\frac{3}{2}-$	1.1
10.52	$\frac{1}{2}+$	0.1	3.92	$\frac{3}{2}+$	0.8	6.80	$\frac{5}{2}+$	0.3
10.56	$\frac{1}{2}+$	0.1	4.30	$\frac{3}{2}-$	2.2	6.80	$\frac{7}{2}-$	2.5
10.60	$\frac{9}{2}-$	1.4	4.30	$\frac{5}{2}+$	1.0	7.60	$\frac{3}{2}-$	1.1
10.64	11/2+	2.1	4.30	$\frac{7}{2}-$	6.4	7.60	$\frac{5}{2}+$	0.2
10.68	$\frac{1}{2}-$	0.2	4.30	$\frac{3}{2}+$	0.7	7.60	$\frac{7}{2}-$	2.3
10.72	$\frac{9}{2}+$	0.7	4.30	$\frac{5}{2}-$	4.5	7.60	$\frac{3}{2}+$	0.1
10.76	11/2−	1.7	4.30	$\frac{7}{2}+$	1.9	7.60	$\frac{5}{2}-$	1.8
10.80	$\frac{3}{2}-$	0.2	4.50	$\frac{1}{2}+$	0.3	7.60	$\frac{7}{2}+$	0.3
10.84	$\frac{5}{2}+$	0.3	4.50	$\frac{1}{2}-$	1.2	7.70	$\frac{1}{2}+$	0.1
10.88	$\frac{7}{2}-$	0.5	5.50	$\frac{1}{2}-$	0.9	7.70	$\frac{9}{2}-$	5.1
10.92	$\frac{3}{2}+$	0.2	5.50	$\frac{1}{2}+$	0.2	7.70	11/2+	1.0
10.96	$\frac{5}{2}-$	0.3	5.80	$\frac{3}{2}+$	0.4	7.70	$\frac{1}{2}-$	0.7
11.00	$\frac{7}{2}+$	0.5	5.80	$\frac{5}{2}-$	2.6	7.70	$\frac{9}{2}+$	0.3
		Sum = 244.7	5.80	$\frac{7}{2}+$	1.1	7.70	11/2−	7.3
			5.80	$\frac{3}{2}-$	1.4	8.30	$\frac{3}{2}+$	0.1
Mg^{23}			5.80	$\frac{5}{2}+$	0.5	8.30	$\frac{5}{2}-$	1.6
ground	$\frac{3}{2}+$	2.0	5.80	$\frac{7}{2}-$	3.7	8.30	$\frac{7}{2}+$	0.2
0.44	$\frac{5}{2}+$	2.7	6.20	$\frac{1}{2}-$	0.7	8.30	$\frac{3}{2}-$	1.0
2.08	$\frac{7}{2}+$	3.6	6.20	$\frac{9}{2}+$	1.2	8.30	$\frac{5}{2}+$	0.1
2.39	$\frac{1}{2}+$	0.5	6.20	$\frac{1}{2}+$	0.2	8.30	$\frac{7}{2}-$	2.1
2.64	$\frac{5}{2}+$	1.5	6.20	$\frac{9}{2}-$	6.6	8.50	$\frac{1}{2}-$	0.4
2.70	$\frac{3}{2}+$	1.1	6.80	$\frac{3}{2}+$	0.2	8.50	$\frac{1}{2}+$	0.0
2.98	$\frac{9}{2}+$	4.0	6.80	$\frac{5}{2}-$	1.9			Sum = 93.4
3.68	$\frac{3}{2}-$	2.5	6.80	$\frac{7}{2}+$	0.6			

the information on level densities available from the $F^{19}+\alpha$ and $Ne^{22}+p$ reactions (allowing in each case for missed levels of high spin). The distribution of these levels in spin and parity, as given in the table, is plausible. The calculations show that the only levels in Mg^{23} and Na^{23} which are fed appreciably are those of very high spin. The distribution of the levels in spin I was assumed to be proportional to

$$(2J+1)\exp[-I(I+1)/2\sigma^2]$$

with a spin cutoff parameter[9] $\sigma=3.5$. In addition the lowest level of spin I was assumed to be at an energy of $0.20[I(I+1)-I_0(I_0+1)]$, where I_0 is the spin of the ground state of $Na^{23}(I_0=\frac{3}{2})$. Without this latter assumption the cross sections to a small number of low-lying high-spin levels of Na^{23} and Mg^{23} are so large

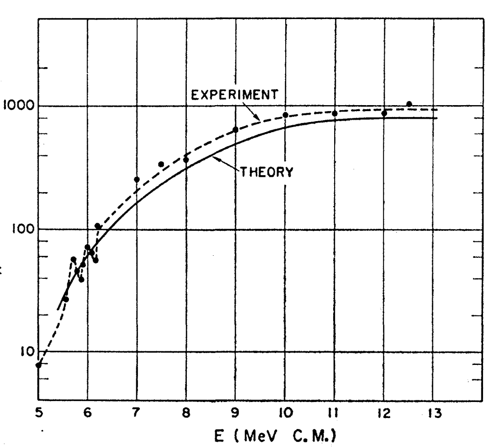

FIG. 2. Comparison of experimental and theoretical absorption cross sections of the $C^{12}+C^{12}$ system as a function of the center-of-mass bombarding energy.

TABLE III. The calculated absorption cross section and the branching ratios for each value of $J\Pi$ of the reactions of the $C^{12}+C^{12}$ system at 11.4-MeV bombarding energy of the center-of-mass.

$J\Pi$	$\sigma_{J\Pi}(\text{abs})/\Sigma_{J\Pi}\sigma_{J\Pi}(\text{abs})$ in %	Branching ratios in %: Alphas	Protons	Neutrons	Be^8	C^{12}
0+	2.3	30.2	41.7	21.6	2.5	4.0
2+	11.6	26.9	45.9	23.5	1.6	2.1
4+	21.1	32.0	44.0	20.4	1.6	2.0
6+	30.2	46.7	35.5	12.0	2.6	3.2
8+	28.6	74.9	12.5	1.0	4.9	6.7
10+	5.9	91.7	0.7	0	3.3	4.2
12+	0.3	97.9	0	0	0.8	1.3

[9] See Sec. 7 below for similar estimates of the spin cutoff parameter.

that the probability of nucleon emission is doubled. Such a large amount of nucleon emission is not in accord with experiment: The assumptions made in connection with Table II yield the prediction that neutrons and protons together contribute about 40% of the total yield from the compound nucleus: The total production of alpha particles is predicted to be approximately equal to the sum of proton and neutron production. This result of the calculation is in reasonable accord with the experimental results.[10]

The low-emission probability of nucleons to low-spin states arises from the high angular momentum imparted to the compound system by the $C^{12}+C^{12}$ pair. To show this, Table III gives the fraction of the $C^{12}+C^{12}$ absorption cross section feeding each value of $J\Pi$ as well as the corresponding branching ratios for each $J\Pi$ for each pair of reaction products. For the low values of $J\Pi$, which are not important in our problem, the nucleon channels predominate in the decay of the compound system, while for the high values of $J\Pi$, alpha particles and heavy ions dominate the decay. The variation of branching ratios with $J\Pi$ clearly demonstrates the need for a theory which considers conservation of J and Π.

5. COMPARISON OF DATA AND THEORY FOR AVERAGE CROSS SECTIONS OF THE $C^{12}(C^{12},\alpha)Ne^{20}$ REACTIONS

The experiments yielding the average cross-section measurements to be compared with the calculations of the preceding section were discussed in the accompanying paper.[1]

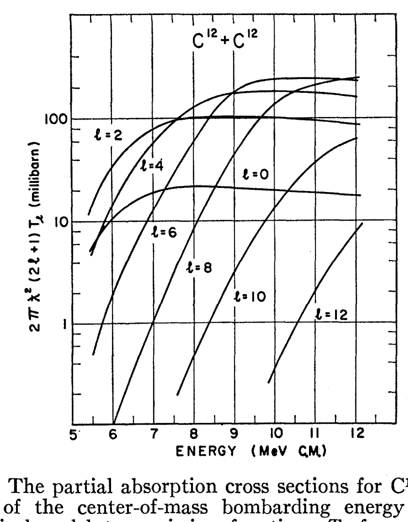

FIG. 3. The partial absorption cross sections for $C^{12}+C^{12}$ as a function of the center-of-mass bombarding energy calculated from optical-model transmission functions T_l for each partial wave l. The optical-model parameters are given in Table I and the total absorption cross section in Fig. 2.

TABLE IV. Experimental and calculated values of angle-integrated cross sections for the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction. α_0 refers to emitted alpha particles leaving Ne^{20} in its ground state, and α_1 to Ne^{20} in its first excited state. The subscript 8 on $\sigma_u(\alpha_0)$ refers to the $J\Pi=8+$ part of the $\sigma(\alpha_0)$ cross section (Ref. 1). The errors in the experimental cross sections are estimated probable errors; the errors in the calculated values are sample-size errors estimated by means of Eq. (33).

	Exp. (mb)	Calc. (mb)
$\sigma(\alpha_0)$	19.2(±4)	16.0±1.5
$\sigma_8(\alpha_0)$	10.5(±2)	8.6±1.8
$\sigma(\alpha_1)$	63.0(±14)	39.3±2.1

The total absorption cross section of the $C^{12}+C^{12}$ system is shown on Fig. 2 as a function of the center-of-mass energy. The optical-model calculations shown on the figure agree well with the data and are a check on the $C^{12}+C^{12}$ transmission functions used in the statistical theory calculations. Figure 3 shows, at each energy, the contributions of various partial waves to the absorption cross section. (Cf. second column of Table III.) Similarly, Fig. 4 gives the contribution of various $J\Pi$ to the $C^{12}(C^{12},\alpha_0)Ne^{20}$ reaction. The contributions of 6+ and 8+ states gain in importance because of the behavior of the branching ratios discussed above.

The experimental and calculated values of the integrated cross section are compared in Table IV. The table lists the mean-integrated cross sections[11] for the

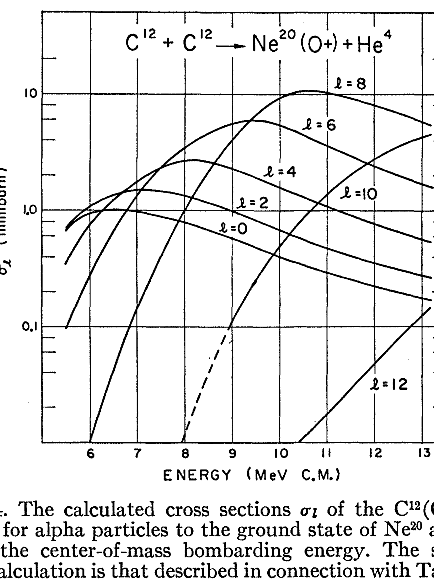

FIG. 4. The calculated cross sections σ_l of the $C^{12}(C^{12},\alpha)Ne^{20}$ reaction for alpha particles to the ground state of Ne^{20} as a function of the center-of-mass bombarding energy. The statistical theory calculation is that described in connection with Table II.

[10] E. Almqvist, J. A. Kuehner, D. McPherson, E. W. Vogt, and J. D. Prentice, in *Proceedings of the Third Conference on Reactions between Complex Nuclei, Asilomar California*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, 1963); E. Almqvist, D. A. Bromley, and J. A. Kuehner, in *Proceedings of Second Conference on Reactions between Complex Nuclei, 1960*, edited by A. Zucker, E. Halbert, and F. T. Howard (John Wiley & Sons, Inc., New York, 1960), p. 282.

[11] The amount of O^{16} contaminant in the C^{12} target is not sufficient to have an appreciable effect on the measured cross section. The amount of contaminant can be ascertained by the strength of the observed $C^{12}(O^{16},\alpha)Mg^{24}$ reaction for which a small ground-state peak is seen. The $C^{12}(C^{12},\alpha_1)Ne^{20}$ reaction is at

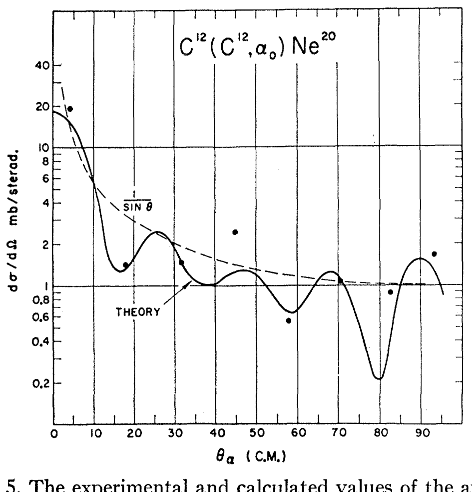

FIG. 5. The experimental and calculated values of the average differential cross section for the $C^{12}(C^{12},\alpha_0)Ne^{20}$ reaction as a function of the center-of-mass angle $\theta_{\alpha 1}$ of the emitted alpha particles. The experimental values are the solid points which have an absolute error of ±20%. The statistical theory curve (solid line) has a finite sample error at each angle of about ±15% [$\sim 0.67\,S^{-1/2}$ with $S=22$]. Neither the experimental points nor the theoretical curve have been multiplied by an arbitrary normalization factor. The angular distribution $(\sin\theta_\alpha)^{-1}$ applicable in the limit of large angular momentum is also shown.

ground state and the first excited state and the partial cross section to the ground state for $J=8$ states only. The experimental values of the average total cross sections were obtained by averaging over the interval 10.15–12.8 MeV while the σ_8 cross sections were averaged over the smaller energy interval of 10.15–11.8 MeV. The effect of the finite size of the interval on the fluctuations was discussed in Ref. 1. The errors quoted in the table (with the theoretical cross sections) correspond to the probable error estimated from (33), with a sample size[1] $S=22$, and with weight coefficients calculated as a by-product of the average cross-section calculations.

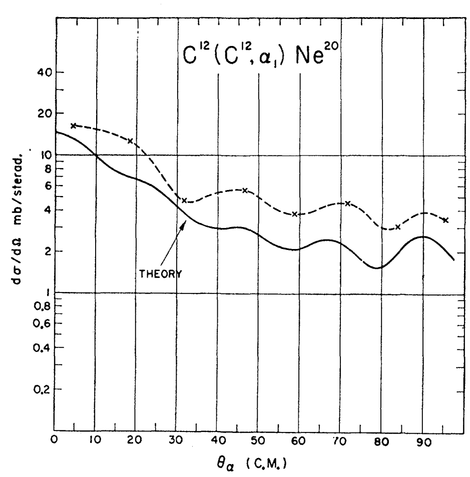

FIG. 6. The experimental and calculated values of the average differential cross section for the $C^{12}(C^{12},\alpha_1)Ne^{20}$ reaction (for alpha particles emitted to the first excited state of Ne^{20}) as a function of the center-of-mass angle θ_α of the emitted alpha particles. The experimental values (crosses joined by an arbitrary broken line) have an error of ±20%. The statistical theory curve (solid line) has a sample error which varies with angle but is roughly ±10% for most angles.

about the same energy as the $C^{12}(O^{16},\alpha_2)Mg^{24}$ and $C^{12}(O^{16},\alpha_3)Mg^{24}$. The population of the latter states relative to the ground state has been calculated with the statistical theory. The resulting correction reduces our α_1 cross sections by a few percent.

The observed average angular distributions are compared to the statistical theory estimates made by means of (3) on Figs. 5 and 6. Neither the data nor the calculations have been renormalized on Figs. 5 and 6 so that the good agreement in both shape and magnitude is of interest. Figure 5 shows how the angular distributions for alpha particles emitted to the ground state of Ne^{20} tend to approach the $(\sin\theta)^{-1}$ behavior appropriate in the classical limit of large angular momentum.[12]

Both the calculated shape and the calculated absolute magnitude of the cross section for alpha particles emitted to the ground state of Ne^{20} are in good agreement with the observations in spite of the fact that the condition (15) does not hold. In fact the agreement is comparable to that for average cross sections involving only nucleons.[13] Since the condition (15) holds for nucleon reactions and not for heavy ions the present results suggest that the usual statistical theory is applicable even for cases for which the transmission functions approach unity. (Figure 5 and Table IV.) The corresponding comparison for the first excited state (Table IV and Fig. 6) does not yield quite such good agreement.

6. ESTIMATE OF LEVEL SPACINGS AND LEVEL WIDTHS

The approximate connection between $\langle \Gamma_c^{J\Pi}\rangle / D^{J\Pi}$ and optical-model transmission functions was given by the second step of (2), based on (19). This connection enables us to calculate

$$\langle \Gamma^{J\Pi}\rangle / D^{J\Pi} = \sum_c \langle \Gamma_c^{J\Pi}\rangle / D^{J\Pi}, \qquad (35)$$

where $\langle \Gamma^{J\Pi}\rangle$ is the average total width and the sum runs over all open channels. By means of (2), the quantities τ_l were evaluated and summed using a G-20 computer to give the results shown in Table V for center-of-mass bombarding energies of 6 and 11.4 MeV. The labels $J\Pi$ are used in Eq. (35) to emphasize that the averages are computed separately for states of each spin and parity in Mg^{24}. Also shown in the table are the experimental values[14] of widths observed for spin-2 and spin-4 states at 19.6 and 19.9 MeV, respectively, in Mg^{24} and the value for spin-8 levels obtained in the accompanying paper.

[12] V. E. Viola, H. M. Blann, T. D Thomas, in *Proceedings of the Second Conference on Reactions between Complex Nuclei, 1960*, edited by A. Zucker, F. T. Howard, and E. C. Halbert (J. Wiley & Sons, Inc., New York, 1960), p. 224.

[13] E. W. Vogt, Phys. Letters **7**, 61 (1963).

[14] E. Almqvist, D. A. Bromley, J. Kuehner, and B. Whalen, Phys. Rev. **130**, 1140 (1963).

TABLE V. Calculated values of $\langle\Gamma\rangle/\langle D\rangle$ and measured values of $\langle\Gamma\rangle_{exp}$ for various $J\Pi$ at two excitation energies in Mg^{24}. The sums $\sum_i$ run over all exit channels available to the compound nucleus.

	$E^*=19.9$ MeV				$E^*=25.3$ MeV			
J	$\sum_i T_i$	$\sum_i \tau_i$	$\langle\Gamma\rangle/D$	$\langle\Gamma\rangle_{exp}$	$\sum_i T_i$	$\sum_i \tau_i$	$\langle\Gamma\rangle/D$	$\langle\Gamma\rangle_{exp}$
0	11.43	19.30	3.12		72.5	147	23.8	
2	28.96	46.90	7.58	130 keV	219	425	68.7	
4	19.01	31.50	5.09	100 keV	177	349	56.4	
6	5.82	9.99	1.62		75.5	160	25.9	
8	1.49	2.03	0.33		21.4	43.3	7.00	120 keV
10	0.12	0.12	0.02		5.54	8.67	1.40	
12					0.71	0.79	0.13	
14					0.025	0.025	0.004	

The total width Γ_0 of a Mg^{24} state at high excitation energies ($\sim$25 MeV) is made up of many partial widths Γ_c, each corresponding to one of the available decay channels shown on Fig. 1. Because the number of decay channels is so large, the total widths do not fluctuate appreciably, and we can equate each individual measured total width to the average total width $\langle\Gamma\rangle$.

$$\Gamma_0=\langle\Gamma\rangle=\sum_c\langle\Gamma_c\rangle.$$

The statistical theory calculations of Tables II and III showed that the nucleon channels, whose properties are not well known, are not expected to contribute a large fraction of the total width of a high-spin level in Mg^{24}. The estimate of the level spacing D from the measured value of Γ by means of Eq. (35) then relies mainly on levels of known spin and parity. These estimates obtained from the data shown in Table V are 17.2 and 19.6 keV for 2+ and 4+ levels at 19.9-MeV excitation in Mg^{24} and 17.1 keV for 8+ levels at 25.3-MeV excitation energy. We now wish to normalize all the values of D to a single excitation energy in Mg^{24} to see the effect of spin on the average level spacing and level width.

The connection between level spacing and excitation energy in the Fermi gas model of the nucleus may be written according to Newton[15]:

$$D(U)=\text{const}AG_Z^{1/2}G_N^{1/2}(2U+3t)^2\times\exp[-2(\pi^2GU/6)^{1/2}], \quad (36)$$

where A is the atomic mass and $G=G_Z+G_N$ is the density of single-particle orbits at the Fermi energy U; the subscripts Z and N refer to protons and neutrons, respectively. Effective values of the quantities G for various nuclei may be obtained from tabulations in papers by Newton[15] and by Cameron.[16] The equivalent energy U of the Fermi gas is obtained by subtracting from the nuclear excitation energy E^*, the pairing correction. The nuclear temperature t is given by

$$t=(6U/\pi^2G)^{1/2}=(U/a)^{1/2}, \quad (37)$$

which also defines the constant "a" in terms of G. We may then rewrite Eq. (36)

$$D(U)=B[2U+3(U/a)^{1/2}]^2\exp[-2(aU)^{1/2}], \quad (38)$$

where "B" and "a" are constants to be determined by experiment for any given nucleus. In taking the ratio of level spacings at two different excitation energies only the constant "a" is required. We shall determine values of this constant and the level spacing (i) from available experimental data on Mg^{24}, and (ii) from the estimates on neighboring nuclei.[15–17] For Mg^{24} the pairing correction is taken to be 4.58 MeV.[16]

(i) The review article by Endt and Van der Leun lists 7 levels with properties 2+ between 11.38 and 13.09 MeV out of 20 natural-parity levels for which assignments are given. Assuming that the same fraction of the ten unassigned levels also have properties 2+, we obtain the average level spacing D_{2+} equal to 180 ($\pm$70) keV at a mean excitation energy of 12.2 MeV. At 19.5 MeV our measurement gives D_{2+} as 17.2 keV. These results taken together yield the value 2.58 for the constant "a"; the corresponding value of the single-particle level density G is 1.61 MeV^{-1}.

(ii) Following the prescription given by Newton,[15] a value of level density G is obtained.

$$G=(0.03772)(\bar{j}_Z+\bar{j}_N+1)A^{2/3} \quad (39)$$
$$=1.88\text{ MeV}^{-1}.$$

Here the effective $\bar{j}$ values for neutrons and protons based on the single-particle level ordering of Klinkenberg[18] are both taken to correspond[15] to $\bar{j}$ equal to $\frac{5}{2}$. The quantity A is the atomic mass; the constant is based on a fit to the observed level densities at the neutron binding energy (i.e., near 6–8-MeV excitation) for 52 nuclei made by Newton.[15] The corresponding value of "a" is computed to be 3.09 to be compared with 2.58 above.

Cameron[16] has estimated the single-particle orbit separations from the differences between the binding energies of successive nucleons and has given a tabulation of estimated values of the level density G for various

[15] T. D. Newton, Can. J. Phys. **34**, 804 (1956).
[16] A. G. W. Cameron, Can. J. Phys. **36**, 1040 (1958).
[17] M. L. Halbert and F. E. Durham, in *Proceedings of the Third Conference on Reactions between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, 1963), p. 223.
[18] P. F. A. Klinkenberg, Rev. Mod. Phys. **26**, 327 (1954).

TABLE VI. Observed and calculated values of the spacing D_2 of 2+ states in Mg^{24} at various excitation energies E^*.

E^* MeV	U MeV	D_2 (obs) keV	D_2 (Calc.) keV (a)	(b)	(c)	(d)	Mean D_2 keV
12.2	7.6	180 (±70)	180[e]	248	125	326	225
19.5	14.9	17.2	17.2[e]	17.2[e]	17.2[e]	17.2[e]	17.2[e]
25.0	20.4		3.60	3.0	5.66	2.45	3.68

[a] Equation (38) with both "a" and "B" determined by fitting to two normalization points.
[b] Newton's prescription (Ref. 15, see text).
[c] Cameron's prescription (Ref. 16, see text).
[d] Halbert and Durham's prescription (Ref. 17, see text).
[e] These values were used for normalization.

nuclei. The energy dependence of G which is implicit in Cameron's empirical procedure is not significant over the range of U of concern here; however, his expressions for nuclear temperature and level densities are not of the simple form of Eqs. (37) and (38) and the constants cannot be directly compared with those previously given. We therefore shall compare the resulting level spacings computed using his prescription with those obtained in other ways. The results are summarized in Table VI. A detailed review of the assumptions and limitations of these various level density formulas and others has been given by Ericson.[19]

Halbert and Durham[17] have studied level densities in $Al_{13}{}^{26}$, $S_{16}{}^{33}$, and $A_{18}{}^{37}$ at excitation energies similar to those of interest here. They fit their data with an expression of the form

$$D(E)=C(E-b)^2 \exp\{-2[a(E-b)]^{1/2}\} \qquad (40)$$

and find the best fit "pairing energy" b equal to zero except for Al^{26}, where it is -1 MeV. Equation (40) is very similar to Eq. (38) when U is substituted for $(E-b)$. In order to make a comparison with the Mg^{24} data, we shall assume that the single-particle level-density G, and hence the constant "a" are both proportional to A as has been suggested by Lang and Le Couteur.[20] The observed values of "a" divided by the corresponding atomic mass are given in Table VII. The estimate of the level spacings of Mg^{24} given by Eq. (40) using mean value 0.158 of the ratio a/A are shown in column (iv) of Table VI. We see that the four

TABLE VII. Comparison of level-density parameters based on Halbert and Durham's analysis (Ref. 17) and various level density formulas used in the text. (a), (b), and (c) give the value of a/A corresponding to the columns of Table VI.

	Halbert and Durham (Ref. 17) (d) Al^{26}	S^{33}	A^{37}	Mean	(a)	(b)	(c)
a/A	0.146	0.167	0.162	0.158	0.108	0.129	0.095

[a] Equation (38) with both "a" and "B" determined by fitting to two normalization points.
[b] Newton's prescription (Ref. 15, see text).
[c] Cameron's prescription (Ref. 16, see text).
[d] Halbert and Durham's prescription (Ref. 17, see text).

[19] T. Ericson, Phil. Mag. Suppl. **9**, 425 (1960).
[20] J. M. B. Lang and K. J. LeCouteur, Proc. Phys. Soc. (London) **A67**, 586 (1954).

different methods give reasonably consistent results for the extrapolation of the observed level spacing from 19.5-MeV excitation to the other energies.

The spacing of spin-2 and spin-4 levels have been normalized to 25.0-MeV excitation using the mean extrapolation described in Table VI. The resulting dependence of level spacing on spin is shown in Fig. 7 to be consistent with a spin cutoff parameter $\sigma=3.5$. Also shown in the figure is the dependence of level width on spin. The "experimental" points for spin 2 and 4 were obtained by combining the computed values $\langle\Gamma\rangle/\langle D\rangle$ in Table IV with the values of $\langle D\rangle$ taken at 25-MeV excitation energy.

7. SPIN CUTOFF PARAMETER AND MOMENT OF INERTIA OF Mg^{24}

The spin dependence of the level spacing in the Fermi gas model is given by

$$D_J=D_0(E)\{(2J+1)^{-1} \exp-[J(J+1)/2\sigma^2]\}, \qquad (41)$$

where the spacing of spin-zero levels $D_0(E)$ depends on the excitation energy E through Eq. (36). J is the level spin and σ is the spin cutoff parameter. The exponent in Eq. (41) is related to the rotational energy, and hence to the moment of inertia I through:

$$E_{\rm rot}=(\hbar^2/2I)\{J(J+1)\} \qquad (42)$$
$$=(t/2\sigma^2)\{J(J+1)\},$$

where t is the nuclear temperature already defined by Eq. (32) in terms of the energy U and the constant "a." From Eqs. (42) and (37), we get

$$(\hbar^2/2I)=U^{1/2}/(2\sigma^2a^{1/2}) \qquad (43)$$
$$=108 \text{ keV},$$

for 25-MeV excitation energy in Mg^{24}. The values of σ and $a^{1/2}$ were taken to be 3.5 and 1.71 from Fig. 7 and Table VII, respectively. It should be noted that the product $[\sigma^2(a)^{1/2}]$ in the denominator of Eq. (43) is less sensitive to the exact value of "a" then is σ^2 itself. This is because the method (described in the previous section) of normalizing the low-spin level spacings to 25 MeV is such that small values of "a" lead to large σ and vice versa.

The value of 108 keV for $\hbar^2/2I$ at 25-MeV excitation is compared in Table VIII with that deduced from the

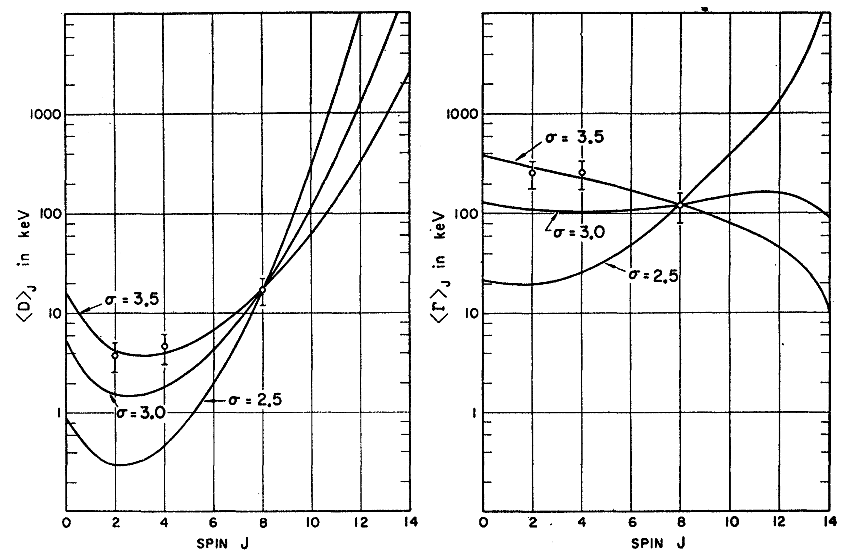

FIG. 7. The average level spacings, $\langle D\rangle_J$ and average total level widths $\langle\Gamma\rangle_J$ obtained from statistical theory transmission functions and from measured total widths for an excitation energy of 25 MeV in Mg^{24}. The observed spacings and widths (points) of the $J=2$, 4 states have been normalized to this energy as discussed in the text. The relative spacings and widths for various values of the spin cutoff parameter are shown by the solid lines.

level spacings of the ground-state rotational band and with the rigid-body value for Mg^{24}. The latter depends both on the radius R and the deformation parameter β through the relation

$$\frac{\hbar^2}{2I_{\text{rigid}}}=\frac{42.7\times10^3}{2\{(25)AR^2(1+0.31\beta+0.44\beta^2+\cdots)\}}\ \text{keV},$$

where A is the mass number and R is in fermis. The value of the deformation parameter β equal to 0.5 is deduced from Coulomb excitation studies[21] of the $E2$ radiation width to the first excited state of Mg^{24} and is consistent with that deduced by minimizing the total energy for all the occupied Nilsson levels.[22] Examination of Table VIII shows that the moment of inertia at 25-MeV excitation is equal to the rigid-body value for the nuclear size and shape used in column three. It seems likely that a somewhat smaller radius parameter would be more realistic, in which case the measurements suggest that the value of the deformation must be somewhat increased since the rigid-body value of the moment of inertial can not be exceeded.

TABLE VIII. Comparison of theoretical and experimental values of $\hbar^2/2I$, where I is the moment of inertia for two excitation energies E^* in Mg^{24}.

Exp at $E^*=25$ MeV	Ground-state rotational band	Rigid-body value $R=1.4A^{1/3}$, $\beta=0.5$	$R=1.2A^{1/3}$, $\beta=0.5$
108	237	108	147

[21] H. E. Gove and C. Broude, in *Proceedings of the Second Conference on Reactions between Complex Nuclei, 1960*, edited by A. Zucker, F. T. Howard, and E. C. Halbert (John Wiley & Sons, New York, 1960), p. 57; I. Kh. Lemberg, *ibid.*, p. 112.

[22] H. E. Gove, in *Proceedings of the International Conference on Nuclear Structure, Kingston*, edited by D. A. Bromley and E. Vogt (University of Toronto Press, Toronto, Canada, 1960), p. 438.

8. CONCLUSIONS

A discussion of the theory of average cross sections for heavy-ion reactions showed that the conventional statistical theory is not necessarily applicable because of the large value of $2\pi\langle\Gamma_{\lambda c}\rangle/D$ for the predominant exit channels. However, a comparison of experimental and theoretical average cross sections of the $C^{12}(C^{12},\alpha)Ne^{20}$ reactions showed generally good agreement between the theory and experiment for both the magnitudes of integrated cross sections and the shapes of differential cross sections. The quantitative comparison of theory and experiment included estimates of the effect of cross-section fluctuations through the finite size of the averaging interval. Like the similar agreement between theory and experiment which was found for the cross-section fluctuations of the same reactions,[1] the success of the conventional statistical theory for the $C^{12}+C^{12}$ reactions suggest that the compound-nucleus mechanism predominates. The variety of measurements compared successfully with the compound-nucleus calculations make the present comparison one of the most comprehensive tests which the compound nucleus has received.

The level spacings obtained from measured total widths and calculated transmission functions agree with other data for Mg^{24}. The level spacings yield accurate values of the spin cutoff parameter which, in turn, yield reasonable values of the moment of inertia of Mg^{24} at high excitation energies.